# Finite-Time Noninertial Adaptive Control

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**The topic of this paper is the synthesis of finite-time adaptive control of a nonstationary nonlinear space vehicle without utilizing information about variations of its parameters and nonlinearities. The algorithms of the adaptive control are established for an arbitrarily chosen aggregation function such that their implementation assures the required trajectory bounds of the vehicle and its error-state system and guarantees the same settling time of the vehicle as that of the reference model that can be optimal in an appropriate sense. Illustrative examples are worked out.**

## Nomenclature

$A$ = the absolute value matrix of a matrix $A$, $A=(\alpha_{ij})$, $|A|=(|\alpha_{ij}|)$

$A_{(\cdot)}$ = an $n\times n$ functional matrix, $A_{(\cdot)}:\mathcal{R}\times\mathcal{R}^n\times\mathcal{R}^n\times\mathcal{R}^q\rightarrow\mathcal{R}^{n\times n}:\mathcal{R}^{n\times n}$

$A_{ci}$ = a known $n\times n$ matrix determining a bound of allowable variations of $A_c$, $A_{ci}:\mathcal{R}\rightarrow\mathcal{R}^{n\times n}$, $i=1,2$

$\mathcal{A}_c$ = a set of all allowable elementwise changeable matrices $A_c$ of the vehicle, $\mathcal{A}_c=\{A_c:A_{c1}(t)\le A_c(t)\le A_{c2}(t),\ \forall t\in\mathfrak{I}_0\}$

$B_{(\cdot)}$ = an $n\times m$ functional matrix, $B_{(\cdot)}:\mathcal{R}\times\mathcal{R}^n\times\mathcal{R}^n\times\mathcal{R}^q\rightarrow\mathcal{R}^{n\times m}$

$B_{(\cdot)i}$ = a known $n\times m$ matrix defining a bound of allowable variation of $B_{(\cdot)}$, $B_{(\cdot)i}:\mathcal{R}\rightarrow\mathcal{R}^{n\times m}$, $i=1,2$

$\mathcal{B}_{(\cdot)}$ = a set of all allowable matrices $B_{(\cdot)}$, $\mathcal{B}_{(\cdot)}=\{B_{(\cdot)}:B_{(\cdot)1}(t)\le B_{(\cdot)}(t)\le B_{(\cdot)2}(t)\}$

$b_v$ = a known $m$ vector associated with the vehicle, $b_v\ge 0$ elementwise

$b_{(\cdot)}$ = an $m$ vector function, $b_{(\cdot)}:\mathcal{R}\times\mathcal{R}^n\rightarrow\mathcal{R}^m$, $b_{(\cdot)}=x_{(\cdot)}$ is allowable and then $m=n$, $b_{(\cdot)}=(b_{(\cdot)1}\,b_{(\cdot)2}\ldots b_{(\cdot)m})^T$

$B_v$ = a known $n\times m$ matrix associated with the vehicle, $B_v\ge 0$ elementwise

$\mathcal{B}_v$ = a set of all allowable nonlinearities of the vehicle, $\mathcal{B}_v=\{b_v:|b_v(t,x_v)|\le b_v+B_v|x_v|,\forall t\in\mathfrak{I}_0,\ \forall x_v\in S_{Av}(t)\}$

$C_{(\cdot)}$ = an $n\times q$ functional matrix, $C_{(\cdot)}:\mathcal{R}\times\mathcal{R}^n\times\mathcal{R}^n\times\mathcal{R}^q\rightarrow\mathcal{R}^{n\times q}$

$C_{ci}$ = a known $n\times q$ matrix determining a bound of allowable variations of $C_c$, $C_{ci}:\mathcal{R}\rightarrow\mathcal{R}^{n\times q}$, $i=1,2$

$\mathcal{C}_c$ = a set of allowable elementwise changeable matrices $C_c$ of the vehicle $\mathcal{C}_c=\{C_c:\ C_{c1}(t)\le C_c(t)\le C_{c2}(t),\ \forall t\in\mathfrak{I}_0\}$

$C^n$ = the collection of all nonempty connected bounded subsets of $\mathcal{R}^n$

$D_v$ = an $n\times r$ functional matrix associated with the vehicle, $D_v:\mathcal{R}\times\mathcal{R}^n\times\mathcal{R}^n\times\mathcal{R}^q\rightarrow\mathcal{R}^{n\times r}$; all elements of $D_v$ are either known or adjustable and, hence, $D_v=D_a$

$e$ = the error state vector, $e\in\mathcal{R}^n$, $e=x_M-x_v$

$i$ = a $q$ vector function defining the governing (command) input, $i:\mathcal{R}\rightarrow\mathcal{R}^q$

$\mathcal{I}$ = the set of all allowable governing inputs, $\mathcal{I}=\{i:\|i(t)\|\le\gamma,\ \ \forall t\in\mathfrak{I}_0\}$, $\gamma\in\mathcal{R}_+$

$M(A)$ = the structural matrix of $A=(\alpha_{ij})$, $M=(\mu_{ij})$

$\mu_{ij}$ $=\begin{cases}0 \text{ iff } \alpha_{ij}\equiv 0\\ 1 \text{ iff } \alpha_{ij}\neq 0\end{cases}$

$S(A)$ = the signum matrix of $A, S(A)=(\operatorname{sgn}\alpha_{ij})$

$\operatorname{sgn}\alpha$ $=|\alpha|^{-1}\alpha$ iff $\alpha\neq 0$ and $\operatorname{sgn} 0=0$

$\operatorname{sat}\alpha$ $=\begin{cases}\alpha, & |\alpha|\le 1\\ \operatorname{sgn}\alpha, & |\alpha|\ge 1\end{cases}$

$S$ = a set valued function, $S:\mathcal{R}\rightarrow C^n$, which determines a time varying set $S(t)$; all time-varying sets are accepted continuous in $t\in\mathfrak{I}_0$ [1,2]

$\partial S(t)$ = the boundary of $S(t)\in C^n$

$\bar{S}(t)$ = the closure of $S(t)\in C^n$

$S_1\times S_2$ = the Cartesian product of $S_1$ and $S_2$, $S_1\times S_2=\{(x,y):x\in S_1, y\in S_2\}$

$S_{A(\cdot)}(t)$ = the set of all allowable states of $(\cdot)$ at $t\in\mathfrak{I}_0$

$S_{F(\cdot)}(t)$ = the set of all allowable states of $(\cdot)$ at $t\in(\mathfrak{I}_0\setminus\mathfrak{I}_s)$, $S_{F(\cdot)}(t)\subseteq S_{A(\cdot)}(t)$

$S_{I(\cdot)}(t_0)$ = the set of all allowable initial states of $(\cdot)$

$S_{L(\cdot)}(t)$ = a subset of $S_{F(\cdot)}(t)$ with the property that $\vartheta(t,x)<\vartheta(t,y)$, $\forall t\in\mathfrak{I}_0$, $\forall x\in\bar{S}_{L(\cdot)}$ $\forall y\in S_{A(\cdot)}(t)\ S_{L(\cdot)}(t)$

$S_{(\cdot)v}(t)$ = is associated with the control system of the vehicle, $S_{(\cdot)v}(t)=\{x_v:x_v=x_M+e, x_M\in S_{(\cdot)M}(t), e\in S_{(\cdot)e}(t)\}$

$T(y)$ = the signum matrix of $y=(y_1\,y_2\ldots y_n)^T$, $T(y)=\operatorname{diag}\{\operatorname{sgn}y_1\ \operatorname{sgn}y_2\ldots\operatorname{sgn}y_n\}$

$t$ = time, $t\in\mathcal{R}$

$t_0$ = an initial moment, $t_0\in\mathcal{R}$

$\mathfrak{I}_0$ = a given (or to be determined) finite-time interval, $\mathfrak{I}_0=[t_0,\ t_0+\tau]$

$\mathfrak{I}_s$ = a known (or to be determined) finite-time interval, $\mathfrak{I}_s=[t_0,\ t_0+\tau_s]$

$\mathfrak{I}_0\ \mathfrak{I}_s$ $=]t_0+\tau_s,\ t_0+\tau[$

$u$ = the control vector function, $u:\mathcal{R}\times\mathcal{R}^n\times\mathcal{R}^n\times\mathcal{R}^q\rightarrow\mathcal{R}^r$, $u=(u_1u_2\ldots u_r)^T$

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$\vartheta$ = an aggregation function used for a system aggregation, $\vartheta:\mathcal{R}\times\mathcal{R}^n\rightarrow\mathcal{R}$; it is accepted that $\vartheta(t,y)\in C^{(1,1)}(\mathcal{R}\times\mathcal{R}^n)$

$(d/dt)\vartheta(t,x_{(\cdot)})$ = the Eulerian derivative of $\vartheta$ along motions of $(\cdot)$

$\nabla\vartheta(t,x_{(\cdot)})$ = $[(\partial\vartheta/\partial x_{(\cdot)1})(\partial\vartheta/\partial x_{(\cdot)2})\ldots(\partial\vartheta/\partial x_{(\cdot)n})]^T$

$\vartheta_{s(\cdot)}(t)$ = $\sup[\vartheta(t,y):y\in\bar{S}_{(\cdot)}(t)]$

$\vartheta_{i(\cdot)}(t)$ = $\inf[\vartheta(t,y):y\in\bar{S}_{(\cdot)}(t)]$

$\vartheta^{\partial}_{i(\cdot)}(t)$ = $\inf[\vartheta(t,y):y\in\partial S_{(\cdot)}(t)]$

$x_{(\cdot)}$ = the state vector of a system determined by $(\cdot)$, $x_{(\cdot)}\in\mathcal{R}^n$, $x_{(\cdot)}=(x_{(\cdot)1}\,x_{(\cdot)2}\ldots x_{(\cdot)n})^T$

$x_{(\cdot)}(t;\,t_0,x_{(\cdot)0};\,i)$ = a motion of $(\cdot)$, which is in $x_{(\cdot)0}$ at $t_0$, provided that $i$ is a governing input of $(\cdot)$ on $\mathfrak{J}_0$; $x_{(\cdot)0}=x_{(\cdot)}(t_0)$

$\|y\|$ = the Euclidean norm of $y\in\mathcal{R}^n$, $\|y\|=(y^Ty)^{1/2}$

$\beta$ = a nonlinear function, $\beta:\mathcal{R}\times\mathcal{R}^n\rightarrow\mathcal{R}$

$\sigma\in\{0,1\}$ = $\sigma=1$ iff $(d/dt)\vartheta(t,x_M)<\psi(t)$, $\forall(t,x_M,i)\in\mathfrak{J}_0\times\mathcal{R}^n\times\mathcal{I}$, holds provided $B_Mb_M\equiv0$; $\sigma=0$ iff $b_M(t,x_M)=x_M$ and $(d/dt)\vartheta(t,x_M)<\psi(t)$, $\forall(t,x_M,i,B_M)\in\mathfrak{J}_0\times\mathcal{R}^n\times\mathcal{I}\times\mathcal{B}_M$

$\tau$ = $\tau\in]0,+\infty[$

$\tau_s$ = $\tau_s\in]0,\tau[$ – the settling time of both the reference model and the vehicle

$\phi$ = the empty set

$\psi$ = a function $\psi:\mathcal{R}\rightarrow\mathcal{R}$ being integrable over $\mathfrak{J}_0$

*Superscripts*

$m,n,p,q,r$ = positive integers

*Subscripts*

$a$ = index denoting matrices with elements being either constant or adjustable parameters of the adaptable controller of the vehicle

$c$ = index denoting matrices with either known or unpredictably changeable parameters of the vehicle

$e$ = index associated with the error-state system

$i,j$ = positive integers

$M$ = index denoting the reference model of the control system of the vehicle; all matrices indexed by $M$ are known

$v$ = index denoting the control system of the vehicle, all matrices indexed by $v$ are decomposable into those indexed by $c$ and $a$, e.g., $A_v(t,\ldots)=A_c(t)+A_a(t,\ldots)$

## Introduction

AIRPLANES, rockets, and space vehicles are required generally to realize their desired motions or, at least, motions that are appropriately close to the desired ones on a finite-time interval.[3] Minimum acceptable errors are prespecified. Hence, a set of all allowable state errors until the settling time and a set of all allowable state errors after the settling time has elapsed are defined.

Space vehicles are nonstationary and nonlinear. In addition, some of their parameters and/or nonlinearities vary unpredictably within certain bounds. Adequate models of space vehicles retain these properties as essential ones.

The severe requirements imposed on motions of space vehicles with described features can be hardly satisfied by classical feedback concepts, which was a reason for the development of adaptive control.[4] Meanwhile, fairly all stability-oriented results on adaptive-control systems as reviewed by Landau,[5,6] Hang and Parks,[7] and Lindorf and Carroll[8] were concerned with synthesis of adaptive control on an infinite time interval. Synthesis problems of finite-time stabilizable control were first considered in Ref. 9 and more recently in Ref. 10.

In this paper, we impose and solve the problem of finite-time adaptive-control synthesis in general. The solution guarantees state errors to be in the prespecified bounds and moreover the same settling time of a real, nonlinear vehicle as that of the model that can be optimal in a given sense. The results are based on the concept of practical stability with the settling time,[2,11] and they enable the reduction of the vehicle-control optimization to that of the reference model. If the settling time of the vehicle is to be minimized it will be achieved by minimizing the model settling time.

## Statement of Problems

The system to be considered is governed by

$$\frac{dx_v}{dt}=A_v(t,\ldots)x_v+B_v(t,\ldots)b_v(t,x_v)+C_v(t,\ldots)i+D_v(t,\ldots)u \tag{1}$$

which describes the control system of the vehicle (the vehicle itself together with its adaptable controller). The parameters and/or nonlinearities of the vehicle may be unpredictably changeable within known bounds. We do not need information about the real form of variations of the parameters and nonlinearities.

With the system (1), we associate its reference model

$$\frac{dx_M}{dt}=A_M(t)x_M+B_M(t,x_M)b_M(t,x_M)+C_M(t)i \tag{2}$$

which can be optimal in an appropriate sense and, hence, nonlinear. For example, the reference model (2) can be optimal in the sense of minimizing the settling time $\tau_s$. We shall establish algorithms, whose implementation assures the same settling time $\tau_s$ of the vehicle. This is accomplished by using the error state system

$$\frac{de}{dt}=A_M(t)e+B_M(t,e+x_v)b_M(t,e+x_v)+[A_M(t)-A_v(t,\ldots)]x_v-B_v(t,\ldots)b_v(t,x_v)+[C_M(t)-C_v(t,\ldots)]i-D_v(t,\ldots)u \tag{3}$$

Notice that the nonlinearities $B_M$ and $b_M$ of the reference model (2) depend on both $e$ and $x_v$ in Eq. (3).

We suppose that the set $S_{Ie}(t_0)$ of all allowable initial error-states is prespecified. Then, for any governing allowable input $i$, $i\in\mathcal{I}$, and any allowable $e_0$, $e_0\in S_I(t_0)$, the error state $e(t)=e(t;t_0,e_0;i)$ should be in the set $S_{Ae}(t)$ of all allowable instantaneous errors on $[t_0,t_0+\mathfrak{J}_s]$ and in $S_{Fe}(t)$ on$]\,t_0+\tau_s,t_0+\tau[$. In essence, we require the practical stability with the settling time $\tau_s$ of Eq. (3).[2]

#### Definition

System (3) is *practically stable with the settling time $\tau_s$ with respect to* $\{t_0,\mathfrak{J}_0,S_{Ie}(t_0),S_{Ae}(t),S_{Fe}(t),\mathcal{I}\}$ iff $e_0\in S_I(t_0)$ and $i\in\mathcal{I}$ imply

$$e(t;t_0,e_0;i)\in\begin{cases}S_{Ae}(t) & \forall t\in\mathfrak{J}_s\\ S_{Fe}(t) & \forall t\in\mathfrak{J}_0\setminus\mathfrak{J}_s\end{cases}$$

Moreover, we require that system (3) possess such a stability property too. Naturally, the reference model (2) is assumed to be practically stable with the settling time $\tau_s$, which is more precisely expressed[2] by assumption 1.

**Assumption 1**

There exist both an aggregation function $\vartheta$ and comparison function $\psi$ such that

$$\int_{t_0}^{t}\psi(\tau)\,\mathrm{d}\tau \le \begin{cases}\vartheta^{\partial}_{iAM}(t)-\vartheta_{s\bar{I}M}(t_0) & \forall t\in\mathfrak{T}_s\\ \vartheta^{\partial}_{iLM}(t)-\vartheta_{s\bar{I}M}(t_0) & \forall t\in(\mathfrak{T}_0\setminus\mathfrak{T}_s)\end{cases}$$

and along motions of Eq. (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t,x_M)<\psi(t)\qquad \forall(t,i,x_M)\in\mathfrak{T}_0\times\mathfrak{J}\times\mathfrak{R}^n$$

Now we can formally, but more precisely, state the problems to be solved.

**Statement of the Problems**

What are algorithms of adaptive control $u$ and control of adaptable parameters so that their implementation guarantees that both a) the system (1) be practically stable with the settling time $\tau_s$ with respect to $\{t_o,\ \mathfrak{T}_0,\ S_{Iv}(t_0),\ \ S_{Av}(t),\ \ S_{Fv}(t),\mathfrak{J}\}$, and b) the error state system (3) be practically stable with the setting time $\tau_s$ with respect tp $\{t_o,\ \mathfrak{T}_0,\ S_{Ie}(t_0),\ S_{Ae}(t),\ S_{Fe}(t),\ \mathfrak{J}\}$ for all allowable variations of parameters and nonlinearities of the vehicle: $\forall(A_c,\ B_c,\ C_c,\ b_v)\in\mathfrak{A}_c\times\mathfrak{B}_c\times\mathfrak{C}_c\times\mathfrak{B}_v$, provided that Assumption 1 holds.

We shall solve these problems in the general setting of nonstationary nonlinear systems and time-varying sets by employing the concept of noninertial adaptive control,[10,12,13] and Assumption 2.

**Assumption 2**

The aggregation function $\vartheta$ and the corresponding sets obey both

$$\vartheta^{\partial}_{iAM}(t)-\vartheta_{s\bar{I}M}(t_0)\le\vartheta^{\partial}_{iAe}(t)-\vartheta_{s\bar{I}e}(t_0),\ \ \forall t\in\mathfrak{T}_s$$

and

$$\vartheta^{\partial}_{iLM}(t)-\vartheta_{s\bar{I}M}(t_0)\le\vartheta^{\partial}_{iLe}(t)-\vartheta_{s\bar{I}e}(t_0),\ \ \forall t\in(\mathfrak{T}_0\setminus\mathfrak{T}_s)$$

## Solutions of the Problems

Noninertial control, which was most often used as a relay control, appeared suitable for solutions of problems of optimal control,[14-16] discontinuous control,[16,17] variable structure control systems,[18] and adaptive control systems.[9,12,19-22] It is also advantageous for the simplicity of its realization by relay devices or other typical nonlinear elements.

In general, control $u{:}\mathfrak{T}\times\mathfrak{R}^n\times\mathfrak{R}^n\times\mathfrak{J}\to\mathfrak{R}^r$ is called *noninertial*. It is realized by static, or so called instantaneous, elements, rather than by dynamic ones. If noninertial control is utilized to adjust adaptable matrices and/or as an adaptive control signal, then it is called *noninertial adaptive control.*[10,13]

General noninertial adaptive control of the vehicle (2) is defined by

$$\begin{aligned}&T[(A_M(t)-A_c-A_a)x_v+\sigma B_M(t,e+x_v)b_M(t,e+x_v)\\ &\quad+(1-\sigma)B_M(t,e+x_v)x_v-(B_c+B_a)b_v\\ &\quad+(C_M(t)-C_c-C_a)i-D_vu]=-T[\nabla V(t,e)]\\ &\quad\forall(A_c,B_c,C_c,b_v,t,x_v,e,i)\in\mathfrak{A}_c\times\mathfrak{B}_c\\ &\quad\times\mathfrak{C}_c\times\mathfrak{B}_v\times\mathfrak{T}_0\times\mathfrak{R}^n\times\mathfrak{R}^n\times\mathfrak{J}\end{aligned}\tag{4}$$

This algorithm requires such a choice of adaptable parameters $(A_a,B_a,C_a,D_a)$ and adaptive control $u$ that for each $t\in\mathfrak{T}_0,x_v\in\mathfrak{R}^n$, $e\in\mathfrak{R}^n$ and $i\ e\ \mathfrak{J}$ and for any allowable value of variable parameters $(A_c,B_c,C_c)$ and nonlinearity $(b_v)$, the sign of the left-hand side of Eq. (4) is equal to the sign of its right-hand side.

Implementation of this algorithm of noninertial adaptive control resolves the problems a) and b) under Assumption 1 and Assumption 2. This is precisely explained by Theorem 1.

**Theorem 1**

*If Assumptions 1 and 2 hold, then implementation of noninertial adaptive control satisfying Eq. (4) implies practical stability with the settling time $\tau_s$ of both i) the system (1) with respect to $\{t_0,\mathfrak{T}_0,S_{Iv}(t_0),S_{Av}(t),S_{Fv}(t),\mathfrak{J}\}$ and ii) the error state system (3) with respect to $\{t_0,\mathfrak{T}_0,S_{Iv}(t_0),S_{Av}(t),S_{Fv}(t),\mathfrak{J}\}$ for all allowable variations of parameters and nonlinearities of the vehicle: $\forall(A_c,B_c,C_c,b_v)\in\mathfrak{A}_c\times\mathfrak{B}_c\times\mathfrak{C}_c\times\mathfrak{B}_v$.*

*Proof.* Notice that Assumption 1, (ii) of Theorem 1, and definition of the sets $S_{(\cdot)v}(t)$ imply (i) of the theorem. Hence, we should prove only (ii). At first, we find that along motions of Eq. (3)

$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t,e)\le\frac{\partial\vartheta}{\partial t}+(\nabla\vartheta)^T[A_M(t)e+(1-\sigma)B_M(t,e+x_v)e]$$

due to Eq. (4). Now, Assumption 1, Eq. (4), and $(i=0)\in\mathfrak{J}$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\vartheta(t,e)<\psi(t),\quad\forall(t,e)\in\mathfrak{T}_0\times\mathfrak{R}^n,\quad\forall i\in\mathfrak{J}$$

Integrability of $\psi$, Assumption 1, and Assumption 2 leads to

$$\vartheta[t,e(t;t_0,e_0;i)]\le\begin{cases}\vartheta^{\partial}_{iAe}(t)-\vartheta_{s\bar{I}e}(t_0), & \forall t\in\mathfrak{T}_s\\ \vartheta^{\partial}_{iLe}(t)-\vartheta_{s\bar{I}e}(t_0), & \forall t\in(\mathfrak{T}_0\setminus\mathfrak{T}_s)\end{cases}$$

$$\forall(e_0,i,A_c,B_c,C_c,b_v)\in S_{Ie}(t_0)\times\mathfrak{J}\times\mathfrak{A}_c\times\mathfrak{B}_c\times\mathfrak{C}_c\times\mathfrak{B}_v$$

which completes the proof of (ii) by referring to Ref. 2.

**Example 1**

We shall synthesize noninertial adaptive control of the system (1) given in a special form by

$$\frac{\mathrm{d}x_v}{\mathrm{d}t}=(1+t)^3\begin{pmatrix}\alpha_c & \beta_c\\ 6 & -16.1\end{pmatrix}x_v+\begin{pmatrix}1 & 0\\ \gamma_c & \gamma_a\end{pmatrix}b_v(t,x_v)$$
$$+\begin{pmatrix}\delta_c & \sin t\\ 0 & \delta_c\end{pmatrix}i+\begin{pmatrix}1\\ 0\end{pmatrix}u$$

Here, $u\in\mathfrak{R}$ is a scalar control. It is known only that the parameters $\alpha_c$, $\beta_c$, $\gamma_c$, $\delta_c$ can arbitrarily vary in the bounds $[-1,\ +1]$. The allowable vector nonlinearity can also arbitrarily vary provided $b_v\in\mathfrak{B}_v$

$$\mathfrak{B}_v=\{b_v{:}b_v(t,0)\equiv 0\qquad b_{vi}(t,x_v)/x_{vi}\in[0,1]$$
$$\forall(t,x_v)\in\mathfrak{T}_0\times\mathfrak{R}^2,i=1,2\}$$

where $\mathfrak{T}_0=[0,9]$. The required settling time of the system may not be greater than $\tau_s=2$.

The reference model is given by

$$\frac{\mathrm{d}x_M}{\mathrm{d}t}=(1+t)^3\begin{pmatrix}-12.1 & 4\\ 6 & -16.1\end{pmatrix}x_M-2(1+t)^4\begin{pmatrix}\mathrm{sat}0.1x_{M1}\\ \mathrm{sat}0.1x_{M2}\end{pmatrix}$$
$$+\begin{pmatrix}2 & \sin t\\ 0 & 1\end{pmatrix}i$$

The set of allowable initial states of the model is $S_{IM}(0)=\{x_M: \|x_M\|<2\}$. The set of allowable instantaneous states on [0,2] is $S_{AM}(t)=\{x_M:\|x_M\|<10/(1+t^2)\}$, of allowable instantaneous states on ]2,9[ is $S_{FM}(t)=\{x_M:\|x_M\|<1/(1+t^2)\}$, and the set of all allowable governing inputs $\mathcal{I}=\{i:\|i\|<20\}$.

The given system and its error state system should be practically stable with respect to $\{0, ]0,9[, S_{Iv}(0), S_{Av}(t), S_{Fv}(t),\mathcal{I}\}$ and $\{0,[0,9[, S_{Ie}(0), S_{Ae}(t), S_{Fe}(t),\mathcal{I}\}$, respectively, where the sets of allowable error states are

$$S_{Ie}(0)=\left\{e:\|e\|<2\right\},\; S_{Ae}(t)=\left\{e:\|e\|<\frac{10}{1+t}\right\}$$

$$S_{Fe}(t)=\left\{e:\|e\|<\frac{1}{1+t^2}\right\}$$

At first, we shall analyze the reference model. Function $\vartheta$, $\vartheta(x_M)=\|x_M\|$, is accepted as a tentative aggregation function of the model. Then, the extremal values of $\vartheta$ on the corresponding sets are

$$\vartheta_{sIM}(0)=2 \qquad \vartheta^{\partial}_{iAM}(t)=\frac{10}{1+t} \qquad \vartheta^{\partial}_{iLM}(t)=\frac{1}{1+t^2}$$

so that $S_{LM}(t)=S_{FM}(t)$ can be accepted. Furthermore,

$$\frac{d}{dt}\vartheta(x_M)<-2(1+t)\Rightarrow\psi(t)$$

$$=-2(1+t)\Rightarrow\int_0^t\psi(t)\mathrm{d}t=-(2t+t^2)$$

holds along motions of

$$\frac{dx_M}{dt}=A_M(t)x_M+B_M(t,x_M)b_M(t,x_M)$$

for every $y\in\mathbb{R}^2$, where

$$B_M(t,x_M)=-2(1+t)^4\mathrm{diag}\{\beta(x_{M1}) \quad \beta(x_{M2})\},$$

$$b_M(t,x_M)=x_M$$

and $\beta(\xi)=(\mathrm{sat}\,0.1\xi)\xi^{-1}$. Hence, the reference model is practically stable with the settling time $\tau_s=2$ with respect to $\{0]0,9[,S_{IM}(0), S_{AM}(t), S_{FM}(t),\mathcal{I}\}$.

Referring to the given system we find that allowable variations of changeable parameters are determined by $(A_c,B_c,C_c)\in\mathcal{A}_c\times\mathcal{B}_c\times\mathcal{C}_c$, where

$$\mathcal{A}_c=\left\{A_c:(1+t)^3\begin{pmatrix}-1 & -1\\ 6 & -16.1\end{pmatrix}\right.$$

$$\left.\leq A_c(t)\leq(1+t)^3\begin{pmatrix}1 & 1\\ 6 & 16.1\end{pmatrix}\right\}$$

$$\mathcal{B}_c=\left\{B_c:\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}\leq B_c(t)\leq\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right\}$$

$$\mathcal{C}_c=\left\{C_c:\begin{pmatrix}-1 & \sin t\\ 0 & -1\end{pmatrix}\leq C_c(t)\leq\begin{pmatrix}1 & \sin t\\ 0 & -1\end{pmatrix}\right\}$$

Now, we can formulate the algorithm of noninertial adaptive control by

$$\mathrm{sgn}\{(1+t)^3[\alpha_c+12.1+2(1+t)\beta(x_{v1}+e_1)]x_{v1}$$

$$+(\beta_c-4)x_{v2}+b_{v1}+(2-\delta_c)i_i+u\}=\mathrm{sgn}\,e_1$$

$$\mathrm{sgn}\{2(1+t)^4\beta(x_{v2}+e_2)x_{v2}+\gamma_c b_{v1}+\gamma_a b_{v2}+(\delta_c-1)i_2\}$$

$$=\mathrm{sgn}\,e_2,\forall(t,x_v,e,i)\in\mathfrak{I}_0\times\mathbb{R}^2\times\mathbb{R}^2\times\mathcal{I}$$

provided these equations hold for any $\alpha_c$, $\beta_c$, $\gamma_c$, $\delta_c$, $b_{v1}$, and $b_{v2}$ equal either to $-1$ or $+1$. Then, they will hold for any $\alpha_c$, $\beta_c$, $\gamma_c$, $\delta_c$, $b_{v1}$, and $b_{v2}$ in $[-1, +1]$ due to the linearity of the last expressions in these variables. Implementation of this algorithm guarantees the required practical stability of both the given system and its error state system with the same settling time $\tau_s=2$ as that of the reference model. Moreover, we are able to evaluate trajectory bounds. Trajectories of the vehicle are in the set $S_{Av}(t)$ for $t\in[0,2]$ and in its subset $S_{Fv}(t)$ for $t\in]2,9[$.

Landau and Courtiol[23] established equivalance between parameter and signal synthesis adaptation provided the plant is linear. Their result will be generalized to nonlinear plants in what follows. In order to achieve it we accept Assumption 3.

#### Assumption 3

Vector function $b_v$ is known. Then, we can establish the Lemma.

#### Lemma

*Under Assumption 3 signal synthesis adaptation can be reduced to parameter adaptation.*

*Proof.* Let $D_a$ and $u$ be adequately partitioned so that $D_a u=D_{a1}u_1+D_{a2}u_2+D_{a3}u_3$ and $D_a u=D_{a1}x_v+D_{a2}b_v(t,x_v)+D_{a3}i$. Then, the problem of synthesis of $D_a u\equiv D_v u$, since $D_a\equiv D_v$, is reduced to that of synthesis of $D_{vi}=D_{ai}$, $\forall i=1,2,3$, which proves the lemma.

Under the lemma we may formally write $A_a$, $B_a$, and $C_a$ instead of $(A_{a1}+D_{a1})$, $((B_a+D_{a2})$, and $(C_a+D_{a3})$, respectively, and, if this has been done, then the vehicle is described by

$$\frac{dx_v}{dt}=[A_c+A_a(t,\dots)]x_v+[B_c+B_a(t,\dots)]b_v(t,x_v)$$

$$+[C_c+C_a(t,\dots)]i \tag{5}$$

Formally, Eq. (5) is obtained from Eq. (1) when $u=0$ is set in Eq. (1). Hence, the error-state system corresponding to Eq. (5) is deduced formally from Eq. (3) for $u=0$.

If the lemma holds, then noninertial adaptive control can be synthesized by applying the following algorithm

$$S[A_M(t)-A_c-A_a]=-T[\nabla\vartheta(t,e)]M(A_a)T(x_v)$$

$$T[\sigma B_M(t,e+x_v)b_M(t,e+x_v)+(1-\sigma)B_M(t,e+x_v)x_v$$

$$-(B_c+B_a)b_v(t,x_v)]=-T[\nabla\vartheta(t,e)]$$

$$S[C_M(t)-C_v-C_a)]=-T[\nabla\vartheta(t,e)]M(C_a)T(i)$$

$$\forall(A_c,B_c,C_c,t,x_v,e,i)\in\mathcal{A}_c\times\mathcal{B}_c\times\mathcal{C}_c\times\mathfrak{I}_0\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{I} \tag{6}$$

This algorithm is simpler than Eq. (4) for implementation but is based on the more stringent structural conditions for adaptable matrices, which is expressed by the use of the structural matrices $M(A_a)$ and $M(C_a)$ of adaptable matrices $A_a$ and $C_a$, respectively.

#### Theorem 2

*If Assumptions 1-3 hold, then implementation of noninertial adaptive control determined by Eq. (6) implies practical stability with the settling time $\tau_s$ of both i) the system (5) with respect to $\{t_0, \mathfrak{I}_0, S_{Iv}(t_0), S_{Av}(t), S_{Fv}(t),\mathcal{I}\}$, and ii) the error state system (3) ($u=0$) with respect to $\{t_0,$*

$\mathfrak{I}_0, S_{Ie}(t_0), S_{Ae}(t), S_{Fe}(t), \mathfrak{G}\}$, *for all allowable variations of parameters:* $\forall(A_c, B_c, C_c) \in \mathcal{A}_c \times \mathcal{B}_c \times \mathcal{C}_c$.

Theorem 2 is proved along the same lines as Theorem 1 for $\sigma = 1$.

**Example 2**

Let us reconsider the reference model adaptive system of Example 1 by assuming that $b_v(t,x_v)$ is known and that $u \in \mathcal{R}^2$ rather than $u \in \mathcal{R}$. The control $u$ is now a vector.

At first we propose $u$ to be defined by

$$u = (D_{a1}\ D_{a2}\ D_{a3})\begin{pmatrix} x_v \\ b_v \\ i \end{pmatrix} \quad D_{ai} = \begin{pmatrix} \delta_{i1} & \delta_{i2} \\ \delta_{i3} & \delta_{i4} \end{pmatrix} \forall i = 1,2,3$$

Here, all $\delta_{ij}$ are adjustable parameters. Since $A_a$, $B_a$, and $C_a$ should compensate unknown variations of $\alpha_c$, $\beta_c$, $\gamma_c$, and $\delta_c$ only we set $\delta_{13} = \delta_{14} = 0$, $\delta_{22} = 0$, $\delta_{32} = \delta_{33} = 0$, so that adaptable matrices are now defined by

$$A_a = \begin{pmatrix} \delta_{11} & \delta_{12} \\ 0 & 0 \end{pmatrix} = D_{a1} \qquad B_a = \begin{pmatrix} \delta_{21} & 0 \\ \delta_{23} & \delta_{24} + \gamma_a \end{pmatrix}$$

$$C_a = \begin{pmatrix} \delta_{31} & 0 \\ 0 & \delta_{34} \end{pmatrix}$$

The sets $\mathcal{A}_c$, $\mathcal{B}_c$, and $\mathcal{C}_c$ are the same as in Example 1. We are looking for adaptation algorithms of adjustable parameters in order to assure the practical stability property of the vehicle and its error-state system with the settling time $\tau_s = 2$ of the reference model. Certainly, we can apply Eq. (4). Meanwhile, we apply the simpler algorithm (6), which requires

$$\mathrm{sgn}[-(1+t)^3(\alpha_c + 12.1) - \delta_{11}] = -\mathrm{sgn}(e_1 x_{v1})$$

$$\mathrm{sgn}[(1+t)^3(4 - \beta_c) - \delta_{12}] = -\mathrm{sgn}(e_1 x_{v2})$$

$$\mathrm{sgn}[(1+\delta_{21})b_{v1}(t,x_v) + 2(1+t)^4$$

$$\mathrm{sat}(0.1x_{v1} + 0.1e_1)] = \mathrm{sgn} e_1$$

$$\mathrm{sgn}[(\gamma_c + \delta_{23})b_{v1}(t,x_v) + (\delta_{24} + \gamma_a)b_{v2}(t,x_v)$$

$$+ 2(1+t)^4 \mathrm{sat}(0.1x_{v2} + 0.1e_2)] = \mathrm{sgn} e_2$$

$$\mathrm{sgn}(\delta_c - 2 + \delta_{31}) = \mathrm{sgn}(e_1 i_1)\mathrm{sgn}(\delta_c - 1 + \delta_{34})$$

$$= \mathrm{sgn}(e_2 i_2) \forall (t,x_v,e,i) \in \mathfrak{I}_0 \times \mathcal{R}^2 \times \mathcal{R}^2 \times \mathfrak{G}$$

to be satisfied for any $\alpha_c$, $\beta_c$, $\gamma_c$, $\delta_c$ equal either to $-1$ or to $+1$, because then Eq. (6) holds due to linearity in these parameters. The value and sign of adaptable parameters are accepted such that the precedent algorithm is satisfied, which guarantees the required practical stability properties with the settling time $\tau_s = 2$.

We can realize greater freedom left for the choice of noninertial adaptive control if Assumption 4 holds.

**Assumption 4**

$b_M(t,x_M) = x_M$ and $b_v(t,x_v) = x_v$.

Then, the following equations define a new noninertial adaptive control algorithm:

$$S[A_c + B_c + A_a + B_a - A_M(t) - B_M(t, e + x_v)]$$

$$= T[\nabla \vartheta(t,e)] M(A_a + B_a) T(x_v)$$

$$S[C_c + C_a - C_M(t)] = T[\nabla \vartheta(t,e)] M(C_a) T(i)$$

$$\forall (A_c, B_c, C_c, t, x_v, e, i) \in \mathcal{A}_c \times \mathcal{B}_c \times \mathcal{C}_c \times \mathfrak{I}_0 \times \mathcal{R}^n \times \mathcal{R}^n \times \mathfrak{G} \quad (7)$$

In this case the structural requirement imposed on adaptable matrices is relaxed because the structural matrix $M(A_a)$ of $A_a$ only is replaced by the structural matrix $M(A_a + B_a)$ of $A_a + B_a$.

**Theorem 3**

*If Assumptions 1-4 hold then implementation of noninertial adaptive control determined by Eq. (7) guarantees practical stability with the settling time $\tau_s$ of both i) the system (5) with respect to $\{t_0, \mathfrak{I}_0, S_{Iv}(t_0), S_{Av}(t)$, $S_{Fv}(t), \mathfrak{G}\}$, and ii) the error-state system (3) ($u = 0$) with respect to $\{t_0, \mathfrak{I}_0, S_{Ie}(t_0), S_{Ae}(t), S_{Fe}(t), \mathfrak{G}\}$ for all allowable variations of parameters:* $\forall(A_c, B_c, C_c) \in \mathcal{A}_c \mathcal{B}_c \times \mathcal{C}_c$

Theorem 3 is proved by following the proof of Theorem 1 for $\sigma = 0$.

**Example 3**

Let the adaptive control system of Example 1 be reconsidered now for $b_v(t,x_v) = x_v$ and $u \in \mathcal{R}^2$. We again define $u$ as in Example 2, but now apply Eq. (7). Hence, the adaptable parameters $\delta_{11}$, $\delta_{12}$, $\delta_{21}$, $\delta_{23}$, $\delta_{24}$, $\delta_{31}$, and $\delta_{34}$ are chosen so that

$$\mathrm{sgn}[(1+t)^3(\alpha_c + 12.1) + 1 + \delta_{11} + \delta_{21}$$

$$+ 2(1+t)^4 \beta(x_{v1} + e_1)] = \mathrm{sgn}(e_1 x_{v1})$$

$$\mathrm{sgn}[(1+t)^3(\beta_c - 4) + \delta_{12}] = \mathrm{sgn}(e_1 x_{v2})$$

$$\mathrm{sgn}(\gamma_c + \delta_{23}) = \mathrm{sgn}(e_2 x_{v1})$$

$$\mathrm{sgn}[\gamma_a + \delta_{24} + 2(1+t)^4 \beta(x_{v2} + e_2)] = \mathrm{sgn}(e_2 x_{v2})$$

$$\mathrm{sgn}(\delta_c + \delta_{31} - 2) = \mathrm{sgn}(e_1 i_1),\ \mathrm{sgn}(\delta_c + \delta_{34} - 1) = \mathrm{sgn}(e_2 i_2)$$

$$\forall (t,x_v,e,i) \in \mathfrak{I}_0 \times \mathcal{R}^2 \times \mathcal{R}^2 \times \mathfrak{G}$$

hold for any $\alpha_c, \beta_c, \gamma_c, \delta_c$ equal either to $-1$ or to $+1$. This algorithm guarantees practical stability with the settling time $\tau_s = 2$ of both the vehicle and the error-state system. Bounds of their trajectories for $t \in [0,2]$ are determined by the boundaries of the sets $S_{Av}(t)$ and $S_{Ae}(t)$, and for $t \in ]2,9[$ by the boundaries of $S_{Fv}(t)$ and $S_{Fe}(t)$, respectively.

## Conclusions

The theory of finite-time noninertial adaptive control of nonstationary nonlinear plants (vehicles) has been developed in the paper. It is advantageous for the following reasons: 1) The reference model may be nonlinear nonstationary optimal control system. 2) Minimization of the settling time of the plant (vehicle) is reduced to minimization of the settling time of the reference model. In general, optimization of the adaptive-control system of the plant (vehicle) is reduced to optimization of its reference model. 3) Implementation of noninertial adaptive control defined by the algorithms established in this paper guarantees the required practical stability property of both the plant and its error-state system with the same settling time as that of the reference model. Moreover, the plant trajectories are assured to be within prespecified bounds. 4) Information about variations of both the plant parameters and nonlinearities is not required for implementation of noninertial adaptive control. 5) The adaptive control algorithms are synthesized for the aggregation function of arbitrary form. They enable usage of the aggregation function of the reference model. 6) Noninertial adaptive control is simply realized by digital computers in particular. 7) Great freedom is left for optimization of the level of the stabilizing noninertial adaptive control, which can be used for increasing the speed of convergence of the motions of the plant-control system to those of the reference model.

It seems possible to broaden the theory developed herein to time-discrete and stochastic adaptive control systems. At this stage of the development of noninertial adaptive control there are imposed certain structural requirements on adaptable matrices, which are common to stability-oriented adaptive-control algorithms as pointed out by Narendra and Kudva.[24]

Following Ref. 24 it appears possible to broaden the theory of finite-time noninertial adaptive control to design of finite-time adaptive observers.

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